The Mismeasurement of Risk

Mark Kritzman and Don Rich

Investors typically measure risk as the probability of a given loss or the amount that can be lost with a given probability at the end of their investment horizons. This view of risk considers only the final result, but investors perceive (or should perceive) risk differently. They are affected by exposure to loss throughout the investment period, not just at its conclusion. We introduce two new ways of measuring risk—within-horizon probability of loss and continuous value at risk—that reveal that exposure to loss is substantially greater than investors normally assume.

> In the long run, we are all dead. Economists set themselves too easy, too useless a task, if in tempestuous seasons they can only tell us that when the storm is long past, the ocean will be flat.
>
> John Maynard Keynes
> *A Tract on Monetary Reform* (1923)

Investors typically measure risk as the probability of a given loss or the amount that can be lost with a given probability at the end of their investment horizon. This view of risk considers only the result at the end of the investment horizon, whether the horizon lasts for one day, one week, one year, or many years. It ignores what might happen along the way. We argue that exposure to loss throughout an investment horizon, not only at its conclusion, is important to investors. We introduce two new ways of measuring risk—within-horizon probability of loss and continuous value at risk (VAR).[1] These new risk measures reveal that exposure to loss is substantially greater than investors normally assume.

We wish to distinguish this problem from the issue of parameter estimation. Financial analysts worry that means and variances used in portfolio formation techniques, such as optimization, are estimated with error. These errors bias the resultant portfolio toward assets for which the mean is overestimated and variance is underestimated, which may lead analysts to invest in the wrong portfolio. Financial analysts also worry that higher moments, such as skewness and kurtosis, are misestimated, in which case, extreme returns occur more frequently in reality than is implied by a lognormal distribution.[2] These estimation errors often lead investors to underestimate the probability of loss and to overestimate the probability of gain.

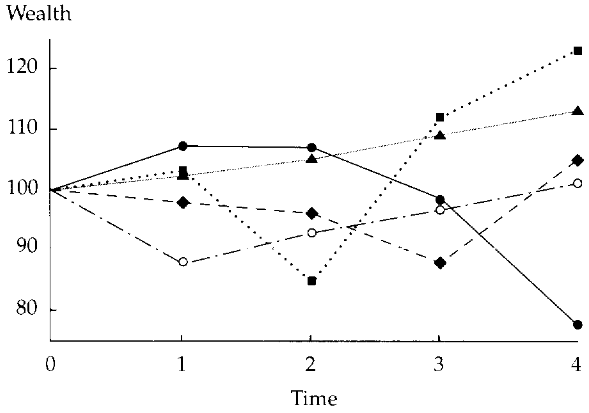
These concerns are legitimate and worthy of investigation. Indeed, the financial industry has made significant advances in addressing these problems. For example, financial analysts who optimize their portfolios often apply compression techniques to reduce sensitivity to estimation error,[3] and they use numerical methods, such as bootstrapping, to capture deviations from theoretical distributions.[4]

Our proposal for risk measurement is not immune to these estimation problems; it suffers from these problems, but it also benefits from the variety of techniques available to improve parameter estimation. Instead of misestimation, we wish to focus on a problem we believe is a more fundamental cause of financial failure: Investors' wealth is affected by risk throughout the period in which it is invested, but risk is generally measured only for the termination of the period. So, even if investors could estimate the moments of a distribution precisely, the investment industry's approach to risk measurement would be inadequate.

Figure 1 illustrates the distinction between risk based on ending outcomes and risk based on outcomes that might occur along the way. Each line represents the path of a hypothetical investment of 100 (dollars, for example) through four periods. The horizontal line at 90 percent represents a loss threshold, which in this example equals 10 percent. Figure 1 reveals that only one of the five paths breaches the loss threshold at the end of the horizon; hence, we might conclude that the likelihood of a 10 percent loss equals 20 percent. However, four of the five paths breach the loss threshold at some point during the investment horizon (and

Mark Kritzman, CFA, is the managing partner of Windham Capital Management, Cambridge, Massachusetts. Don Rich is senior consultant at State Street Associates and an associate professor at Northeastern University, Boston, Massachusetts.

Figure 1. Risk of 10 Percent Loss: Ending Wealth versus Interim Wealth



three of the four paths subsequently recover). So, if we care about the investment's performance along the way to the end of the horizon, we will conclude that the likelihood of a 10 percent loss equals 80 percent, not 20 percent.

Some argue that calculation of daily VAR measures a strategy's exposure to loss within an investment horizon, but it does not. Knowledge of the VAR on a daily basis does not reveal the extent to which losses may accumulate over time. Moreover, even if daily VAR is adjusted to account for prior gains and losses, the investor still has no way to know at the inception of the strategy, or at any other point, the cumulative VAR to any future point throughout the horizon, including interim losses that later recover.

We propose that investors consider exposure to loss throughout the investment horizon. Otherwise, their wealth may not survive to the end. For that purpose, we introduce two innovations to risk measurement—within-horizon probability of loss and continuous VAR. We argue that these innovations capture the nature of investment risk more realistically than current end-of-period measures.

Why Investors Care about Interim Risk

Investors care about exposure to loss throughout their investment horizon, not only at its conclusion, because they often have thresholds that cannot be breached if the investment is to survive to the end of the horizon. Even if survival is not an issue, investors may be motivated to pay attention to within-horizon risk because they will be penalized if they breach a barrier. And finally, if penalization is not an issue, investors often care about results throughout their horizon period for the same reasons that cause them to worry about results at the end of the horizon. Some examples of why investors do or should pay attention to exposure to loss throughout their investment horizon follow.

- *Asset management.* An asset manager is awarded a mandate subject to the provision that the portfolio not depreciate more than 10 percent. Furthermore, the client informs the asset manager that the investment horizon is five years. Should the manager assume that the client will review her performance only once and that this review will occur at the end of the last day of the fifth year? Not likely. A more reasonable expectation is that the client will review performance throughout the five-year horizon and terminate that manager if at any point the portfolio dips below 90 percent of its value at inception. If this asset manager wishes to limit the likelihood of termination to 1 percent, she needs to constrain VAR to 10 percent of the portfolio's initial value not on the basis of the distribution of its ending value nor the distribution of its daily value but, rather, on the distribution of its value throughout the entire investment period.
- *Hedge-fund solvency.* A hedge-fund manager, taking comfort in the belief that the likelihood of a significant loss at the end of the specified investment horizon is remote, leverages his portfolio to increase expected return. His comfort is misguided, however, because a significant decline from the value of the underlying assets from inception to any point during the investment horizon is much more likely than the likelihood implied by the ending distribution of the hedge fund's assets. And a significant interim loss could trigger withdrawals that, added together, might impair the hedge fund's solvency.
- *Loan agreement.* A borrower is required to maintain a particular level of reserves as a condition of a loan. If the reserves fall below the required balance, the loan is called. Again, the probability that this covenant will be breached depends on the distribution of the value of the reserves not at maturity nor on a daily basis but throughout the term of the loan.
- *Securities lending.* Many institutional investors, such as pension funds, lend their securities to other investors who engage in short selling. These investors are required to deposit collateral with the custodian of the securities. The required level of collateral is typically adjusted on a daily basis to offset changes in the values of the securities on loan. Suppose the investor wishes to estimate the amount of additional

collateral that might be required at a given probability for the duration of the loan. This value depends on the distribution of the securities' values throughout the term of the loan.

- *Regulatory requirements.* A bank is required to maintain a capital account equal to a certain fraction of its loan portfolio. A breach of this capital requirement will result in a fine. The probability that the bank will need to replenish the capital account to avoid a breach depends on the distribution of the ratio of the capital account to the loan portfolio throughout the planning horizon, not at the end of the horizon or a finite period within the horizon.

These examples are but a few of the many circumstances in which investors should pay attention to probability distributions that span the duration of their investment horizons. Unfortunately, the common approach for estimating probability of loss and VAR is to focus only on the terminal distribution. In the next section, we present a formula for estimating probability of loss and VAR continuously throughout the investment horizon.

Within-Horizon Exposure to Loss

We estimate probability of loss, Pr_E, at the end of the horizon by (1) calculating the difference between the cumulative percentage loss and the cumulative expected return, (2) dividing this difference by the cumulative standard deviation, and (3) applying the normal distribution function to convert this standardized distance from the mean to a probability estimate, as shown in Equation 1:

$$Pr_E = N\frac{\ln(1+L) - \mu T}{\sigma\sqrt{T}}, \quad (1)$$

where

$N[\bullet]$ = cumulative normal distribution function

L = cumulative percentage loss in periodic units

μ = annualized expected return in continuous units

T = number of years in horizon

σ = annualized standard deviation of continuous returns

The process of compounding causes periodic returns to be lognormally distributed. The continuous counterparts of these periodic returns are normally distributed, which is why the inputs to the normal distribution function are in continuous units.

When VAR is to be estimated, we turn this calculation around by specifying the probability and solving for the loss amount, as shown:

$$VAR = -(e^{\mu T - Z\sigma\sqrt{T}} - 1)W, \quad (2)$$

where

e = base of natural logarithm (2.71828)

Z = normal deviate associated with chosen probability

W = initial wealth

Both of these calculations pertain only to the distribution of values at the end of the horizon and, therefore, ignore variability in value that occurs throughout the horizon. To capture within-horizon variability, we use a statistic called "first-passage time probability."[5] This statistic measures the probability, Pr_W, of a first occurrence of an event within a finite horizon. It is equal to

$$Pr_W = N\frac{\ln(1+L) - \mu T}{\sigma\sqrt{T}} + N\frac{\ln(1+L) + \mu T}{\sigma\sqrt{T}}(1+L)^{2\mu/\sigma^2}. \quad (3)$$

Equation 3 gives the probability that an investment will depreciate to a particular value over some horizon during which it is monitored continuously.[6] Note that the first part of Equation 3 is identical to Equation 1 for the end-of-period probability of loss. It is augmented by another probability multiplied by a constant, and in no circumstances is this constant equal to zero or negative. Therefore, the probability of loss throughout an investment horizon must always exceed the probability of loss at the end of the horizon. Moreover, within-horizon probability of loss rises as the investment horizon expands in contrast to end-of-horizon probability of loss, which diminishes with time.

This effect introduces an interesting twist to the time diversification debate. Samuelson (1963) argued against time diversification by demonstrating that, although probability of loss at the end of the horizon decreases as the horizon grows, this benefit is offset by the increasing magnitude of potential loss. Our result presents a new challenge to those who argue that time diversifies risk because the result does not depend on magnitude of loss. It shows that risk increases even if investors care only about the probability of loss.

We can also use Equation 3 to estimate continuous VAR. Whereas VAR measured conventionally gives the worst outcome at a chosen probability *at the end* of an investment horizon, continuous VAR gives the worst outcome at a chosen probability from inception *to any time during* an investment horizon.

We cannot solve for continuous VAR analytically. We must resort to numerical methods. We set Equation 3 equal to the chosen confidence level and solve iteratively for L. Continuous VAR equals $-L$ times initial wealth.

Applications

In this section, we present applications of within-horizon probability of loss and continuous VAR to currency hedging and a leveraged hedge fund.

Currency Hedging. Suppose we allocate a portfolio equally to Japanese stocks and bonds, represented, respectively, by the MSCI Japan Index and the Salomon Brothers Japanese Government Bond Index. **Table 1** shows, based on monthly returns from January 1995 through December 1999, the standard deviations and correlations of these indexes together with the risk parameters of the Japanese yen from a U.S. dollar perspective.

Table 1. Risk Parameters: Japanese Stocks and Bonds, 1995–99

A. Standard deviation

Asset	Standard Deviation
Stocks	21.87%
Bonds	16.01
Yen	14.81

B. Correlation

Asset	Stocks	Bonds	Yen
Stocks	100.00%		
Bonds	42.01	100.00%	
Yen	53.58	91.08	100.00%

Let us assume further that the underlying portfolio has an expected return of 7.50 percent, hedging costs equal 0.10 percent, and our risk aversion equals 1.00.[7] Based on these assumptions, the optimal exposure to a Japanese yen forward contract is –87.72 percent.[8] The expected return and risk of the unhedged and hedged portfolios are shown in **Table 2**.

Table 2. Expected Return and Risk

Measure	Unhedged	87.72% Hedged
Expected return	7.50%	7.41%
Standard deviation	16.04	9.17

Now, let us estimate the probability of loss for the unhedged and hedged portfolios. **Table 3** shows the likelihood of a 10 percent or greater loss over a 10-year horizon at the end of the horizon and at any point from inception throughout the horizon for an unhedged and optimally hedged portfolio of Japanese stocks and bonds.

Table 3. Probability of Loss: 10-Year Horizon

	10 Percent or Greater Loss		25 Percent or Greater Loss	
Type	End of Horizon	During Horizon	End of Horizon	During Horizon
Unhedged	6.29%	54.14%	2.75%	17.98%
Hedged	0.18	13.91	0.02	0.45

If we were concerned only with the portfolio's performance at the end of the investment horizon, we might not be particularly impressed by the advantage offered by hedging. It reduces the likelihood of a loss equal to 10 percent or greater from 6.29 percent to 0.18 percent, and 6.29 percent might not seem like a large risk. But if, instead, we care about what might happen along the way to the end of the horizon, the advantage of hedging is much more apparent.

Even with the foreknowledge that we are more likely than not at some point to experience a 10 percent cumulative loss, we may consider such a loss tolerable. But what about a loss of 25 percent or greater? Again, calculating the probabilities indicates that, although the impact of hedging on end-of-period outcomes is unremarkable, it vastly reduces the probability of a 25 percent or greater loss during the investment horizon. Although many investment programs might be resilient to a 10 percent depreciation, they are less likely to experience a decline of 25 percent or more without consequences.

Now, let us compare VAR measured conventionally with continuous VAR for the hedged and unhedged portfolios. **Table 4** reveals that the improvement from hedging is substantial whether VAR is measured conventionally or continuously. For example, measured conventionally, hedging improves VAR from a 5 percent chance of no worse than a 14.68 percent loss to a 5 percent chance of no worse than a 26.52 percent *gain*.[9] More important, however, is the substantial difference between VAR measured conventionally and VAR measured continuously. Continuous VAR is more than twice as high as conventional VAR for the unhedged portfolio, and when the portfolio is hedged, continuous VAR shows a substantial loss compared with a substantial gain when it is measured conventionally.

Table 4. VAR (5 Percent): 10-Year Horizon

Type	Conventional VAR	Continuous VAR
Unhedged	+14.68%	+38.68%
Hedged	–26.52	+14.77

Leveraged Hedge Fund. Now consider the implications of these risk measures on a hedge fund's exposure to loss. Suppose we are interested in a hedge fund that uses an overlay strategy, which has an expected incremental return of 4 percent and an incremental standard deviation of 5 percent. This hedge fund also leverages the overlay strategy. **Table 5** shows the expected returns and risks of the hedge fund and its components for varying degrees of leverage.

The data in Table 5 assume that the underlying asset is a government note with a maturity equal to the specified three-year investment horizon and that its returns are uncorrelated with the overlay returns. Managers sometimes have a false sense of security because they view risk as annualized volatility, which diminishes with the duration of the investment horizon, but as we have noted, the fund's assets may depreciate significantly during the investment horizon. **Figure 2** compares the likelihood of a 10 percent loss at the end of the three-year horizon with its likelihood at some point within the three-year horizon for various leverage factors (e.g., 2 to 1). Figure 2 reveals that the chance of a 10 percent loss at the end of the horizon is low but there is a much higher probability that the fund will experience such a loss at some point along the way, which could trigger withdrawals and threaten the fund's solvency.

The same issue applies if exposure to loss is perceived as VAR. **Figure 3** shows the hedge fund's VAR for various leverage factors measured conventionally and continuously. Whereas conventional VAR for leverage factors less than 6 to 1 is negative (a gain) and still very low for leverage factors up to 10 to 1, continuous VAR ranges from approximately 10 percent of the portfolio's value to approximately 40 percent of its value.

Conclusions

Investors measure risk incorrectly if they focus exclusively on the distribution of outcomes at the end of their investment horizons. This approach to risk measurement ignores intolerable losses that might occur throughout an investment period, either as the result of the accumulation of many small losses or from a significant loss that later (too late, perhaps) recovers.

To address this shortcoming, we have introduced two new approaches to measuring risk—measuring within-horizon probability of loss and measuring continuous VAR. Our applications of

Table 5. Leveraged Hedge-Fund Expected Return and Risk

			Leverage				
Measure	Underlying Asset	Overlay Strategy	2	4	6	8	10
Expected return	3.50%	4.00%	11.50%	19.50%	27.50%	35.50%	43.50%
Standard deviation	3.00	5.00	10.44	20.22	30.15	40.11	50.09

Figure 2. Probability of 10 Percent Loss: Three-Year Horizon

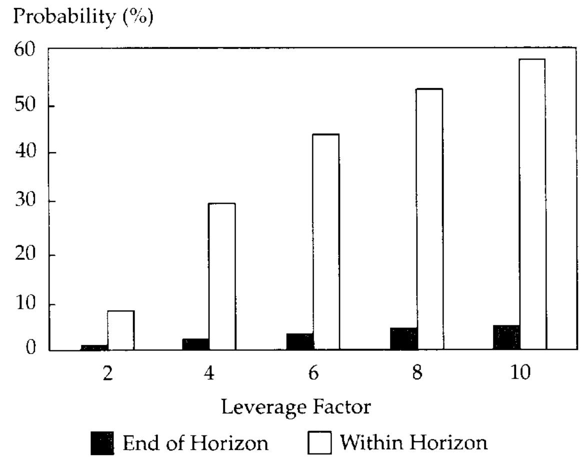
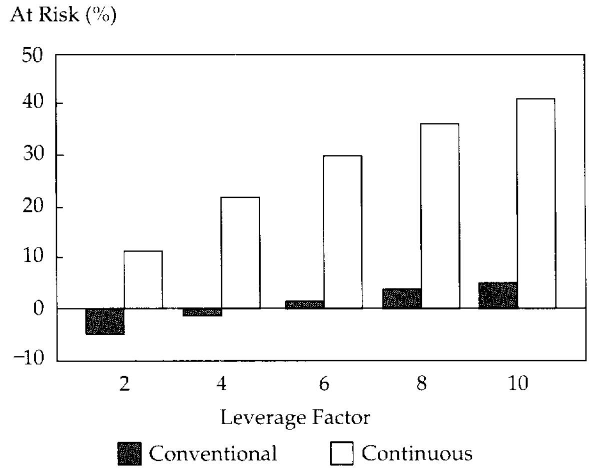


Figure 3. VAR (5 Percent): Three-Year Horizon



these measures in reasonable scenarios illustrates vividly that investors are exposed to far greater risk throughout their expected investment periods than end-of-horizon risk measures indicate.

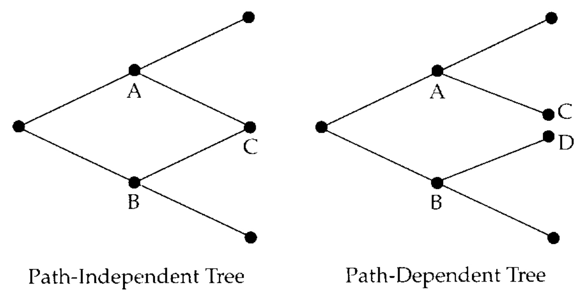
If investors (and their managers) do not care about within-horizon risk, they should. At the very least, investors need to be aware of the likelihood that they will (or will not) be able to sustain their investment strategies. Awareness does not necessarily mean that investors should simply reduce risk, although such a course of action may be warranted. Indeed, if investors are informed of their within-horizon exposures to loss, then if an unpleasant loss occurs, they will not be unduly surprised and will not act to reduce risk out of a misguided perception that the nature of their investment strategy has changed.

Appendix A. Within-Horizon Risk

We present here, first, an intuitive derivation of our risk measures, under the restrictive assumption that expected return equals zero, followed by a more formal derivation in which we relax this assumption.

Intuitive Description. The probability of loss at the end of a horizon is path independent; its estimate does not depend on the particular sequences of returns that lead to the final distribution of outcomes. In contrast, the estimate of probability of loss from inception to any point along the way is path dependent. We cannot ignore the specific sequences of returns that lead to the ending distribution because some sequences that end up above the chosen barrier will at some point along the way have breached it. The distinction between path-independent and path-dependent probabilities is clear from a comparison, as shown in **Figure A1**, of the recombining binomial tree with a tree that does not recombine.

Figure A1. Path-Independent and Path-Dependent Binomial Trees



Path independence is a common assumption in financial analysis because path-independent problems are much easier to solve than path-dependent problems. For example, a binomial tree with 30 time steps has 31 possible terminal outcomes if it recombines (is path independent), whereas it has more than one billion possible outcomes at the final time step (2^{30} = 1,073,741,824) if it does not recombine (is path dependent). One need not be a cynic to assume that financial analysts often sacrifice the realism of path dependency for the sake of computational convenience.

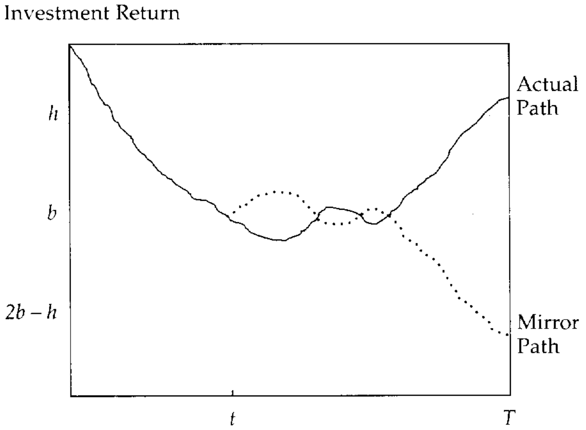
Path-dependent probability. We show that path-dependent probabilities for which the expected return is zero can be restated as simple path-independent probabilities through use of the "reflection" principle (also called "method of images") and then solved with well-known techniques. The reflection principle states that as long as the investment's expected return is zero, for every return path (in continuous units) within any time interval, there is an equally likely return path that is its mirror image.

This principle can be used to solve for another useful estimate—the probability that an investment will depreciate to a particular level at any time during the horizon and then recover. We present a simple procedure for estimating these probabilities (again under the assumption that the investment's expected return equals zero). Indeed, it is easier to present the intuition behind the joint probability that an investment will first breach a barrier and then appreciate to a hurdle than it is to explain the independent probability that an investment will breach a barrier.

Figure A2 shows how we can use the reflection principle to solve for this joint probability. The solid line represents the cumulative-return path (in continuous units) of a hypothetical investment through time. It first depreciates through a hurdle, h, and continues lower through a barrier, b, after which it recovers to a point above the hurdle. To estimate the probability that an investment (with an expected return of zero) will recover to a hurdle after it has breached a barrier, we invoke the reflection principle with its mirror image of this path. Therefore, we trace out a reflected return path (represented by the dotted line in Figure A2) from time t, when the barrier is first breached, to the end of the horizon, time T.

It is straightforward to estimate the path-independent probability that an investment's cumulative return in continuous units will be less than or equal to a particular value (in this case, $2b - h$). With the assumption that continuous returns are normally distributed, this probability equals

Figure A2. The Reflection Principle
($\Pr\{\min R[t] < h\} = \Pr\{R[T] < 2b - h\}$)



$N[(r/n)/\sigma]$, where N is the cumulative normal distribution function, r is the continuous cumulative return, n is the number of years in horizon T, and σ is the standard deviation of the investment's continuous returns.

The path-independent probability of return less than or equal to value $2b - h$ that is below barrier b is identical to the path-*dependent* probability of first crossing the barrier and then declining to value $2b - h$ because $2b - h$ cannot be reached without first crossing b at some point.

The reflection principle guarantees that the probability of achieving a cumulative return equal to h from time t to time T is the same as the probability of achieving a cumulative return equal to $2b - h$ over the same time interval. Therefore, the path-dependent probability of first breaching barrier b at any time t and then recovering to hurdle h at the end of the horizon T is equal to the path-independent probability of experiencing a cumulative return equal to $2b - h$ from inception through time T.

To summarize:

- The path-independent probability of achieving a cumulative return below a barrier over the full investment horizon equals the path-dependent probability of first breaching the barrier at any point during the horizon and then experiencing the balance of the cumulative loss.
- The reflection principle states that for any given return path, as long as expected return is zero, there is an equal probability of a return path that is its mirror image.
- Therefore, the probability of first breaching a barrier and then recovering to a hurdle equals the probability of first breaching a barrier and then falling further to the mirror image of the hurdle, which in turn, equals the probability of experiencing a cumulative path-independent loss equal to the mirror image of the hurdle.

Probability of breaching a barrier. Next, we demonstrate how to estimate the probability that an investment's cumulative return will breach a barrier at least once during an investment horizon. We restate this probability as 1 minus the probability that the investment will never breach the barrier. This statement can be written as 1 minus the joint probability that (1) the cumulative return as of the end of horizon T will exceed the barrier and (2) the minimum cumulative return during the horizon will exceed the barrier.

Next, we invoke the law of total probability: The joint probability of A and B equals the probability of A minus the joint probability of A and not B:

$$\Pr(A, B) = \Pr(A) - \Pr(A, \text{not } B).$$

This probability can be restated as 1 minus the probability that the cumulative return at time T will exceed the barrier, $\Pr(A)$, plus the joint probability that the cumulative return at time T will exceed the barrier and the minimum cumulative return during the horizon will be less than or equal to the barrier, $\Pr(A, \text{not } B)$.

The probability that the cumulative return at time T will exceed the barrier is equal to $N[(b/n)/\sigma]$. As we demonstrated earlier, the joint probability that (1) the cumulative return at time T will exceed the barrier and (2) the minimum cumulative return during the horizon will be less than or equal to the barrier equals $N[(r/n)/\sigma]$. Thus, the probability that an investment's cumulative return will breach a barrier at least once during an investment horizon equals $1 - N[(b/n)/\sigma] + N[(r/n)/\sigma]$.

Once again, to summarize:

- The probability that the minimum cumulative return will be less than the barrier return at some point during the horizon equals 1 minus the probability that the minimum cumulative return is always greater than or equal to the barrier return,
- which equals 1 minus the probability that the cumulative terminal return is greater than or equal to the barrier return and that the minimum return during the horizon is greater than or equal to the barrier return,
- which equals 1 minus the probability that the cumulative terminal return is greater than or equal to the barrier return plus the probability that the cumulative terminal return is greater than or equal to the barrier return and that the minimum cumulative return during the horizon is less than the barrier return.

Formal Description. In this formal description, we start with recovery probability. Suppose we are interested in the joint probability that the value of our portfolio at time T, $S(T)$, will exceed hurdle level H and fall below barrier level B at some point during the investment period, as expressed in the following:

$$\begin{aligned}\Pr[S(T) \geq H, \min S(t) \leq B] &= \Pr[R(T) \geq h, \min R(t) \leq b] \\ &= \left(\frac{B}{S}\right)^{2\mu/\sigma^2} N\left[\frac{\ln(B^2/SH) + \mu T}{\sigma\sqrt{T}}\right].\end{aligned} \quad \text{(A1)}$$

The first equality results from converting periodic units to continuous units [$h = \ln(H/S)$, $b = \ln(B/S)$, $R(T) = \ln[S(T)/S]$, and $R(t) = \ln[S(t)/S$, where $S(t)$ is the value of the portfolio at time $t \in (0,T)$]. Note that $\Pr[R(t) \leq b] \equiv \Pr(\tau \leq T) \equiv$ the global minimum[t:$R(t) = b$, $t \in (0,T)$], where τ is the first-passage time of $R(t)$ to level b. The second equality follows from the reflection principle, a heuristic approach for solving path-dependent problems. The strong Markov property is required for a rigorous proof, but the reflection principle has the advantage of intuition and requires no complex calculations or knowledge of continuous-time mathematics. It is limited, however, because it applies only to cases in which expected return is zero. Nevertheless, we can generalize the case of zero drift to cases with a nonzero drift by applying Girsanov's theorem.[10]

The derivation of the first-passage probability formula requires several algebraic steps to restate it in a more convenient form:

$$\begin{aligned}&\Pr[\min S(t) \leq B] \\ &\quad = \Pr[\min R(t) \leq b] \\ &\quad = 1 - \Pr[\min R(t) \geq b] \\ &\quad = 1 - \Pr[R(T) \geq b, \min R(t) \geq b] \\ &\quad = 1 - \Pr[R(T) \geq b] + \Pr[R(T) \geq b, \min R(t) \leq b] \\ &\quad = \Pr[R(T) \leq b] + \Pr[R(T) \geq b, \min R(t) \leq b] \\ &\quad = N\left[\frac{\ln(B/S) - \mu T}{\sigma\sqrt{T}}\right] + \left(\frac{B}{S}\right)^{2\mu/\sigma^2} N\left[\frac{\ln(B/S) + \mu T}{\sigma\sqrt{T}}\right].\end{aligned} \quad \text{(A2)}$$

The first line converts periodic units to continous units. The second line holds because "proper" probabilities must sum to 1. The third equality appends a redundant condition to the path-dependent probability. It requires that the minimum return always exceed level b, which implies that the terminal return also must exceed level b. This redundant condition facilitates the use of the law of total probability to restate the problem as shown in the fourth line. Terms are collected in the fifth line. The fifth line contains two probability expressions: $\Pr[R(T) \leq b]$, which is a normal path-independent probability, and $\Pr[R(T) \geq b, \min R(t) \leq b]$, which is a recovery probability in which $h = b$. The first probability expression is evaluated directly, and the second probability is evaluated from the first result in this appendix (Equation A1).

Notes

1. Although we discuss this innovation as two risk measures, we acknowledge they are interconnected in such a way that they could be considered two sides of the same coin. This fact is clear from the mathematical description of these risk measures that we present. Although the risk measures are closely related, however, they are not redundant. Investors sometimes care about the relatively high probabilities of moderate losses and, in other instances, are more concerned with extreme events, which are better captured by VAR.
2. If periodic returns are independent and identically distributed, it follows mathematically that they are distributed lognormally. The positive skewness of lognormality results from the process of compounding.
3. A common procedure to reduce sensitivity to estimation error in optimization is to compress the moments for each of the component assets toward the grand means of these moments.
4. One method is "block bootstrapping," in which the analyst samples sequences of contiguous observations with replacement to capture the effect of serial correlation on a sample's distribution.
5. The first-passage probability is described in Karlin and Taylor (1975).
6. In Appendix A, we present an intuitive derivation of our risk measures under the restrictive assumption that expected return equals zero, together with a more formal derivation in which we relax this assumption.
7. A risk aversion equal to 1.00 implies that we are willing to incur 1 unit of hedging cost to lower our portfolio's variance by 1 unit.
8. The forward contract exposure is derived by maximizing portfolio expected return minus risk aversion times portfolio variance, as a function of exposure to the forward contract exposure.
9. A gain appears as a negative value in Table 4 because VAR typically refers to a loss.
10. For a discussion of Girsanov's theorem, see Cox and Miller (1965, pp. 220–221).

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